

Effects of introduction of new resources and fragmentation of existing resources on limiting wealth distribution in asset exchange models.

M. Ali Saif¹

Department of Physics, University of Pune, Ganeshkhind, Pune, 411 007, India

Prashant M. Gade²

Center for Modeling and Simulation, University of Pune, Ganeshkhind, Pune, 411 007, India

Abstract

Pareto law, which states that wealth distribution in societies have a power-law tail, has been a subject of intensive investigations in statistical physics community. Several models have been employed to explain this behavior. However, most of the agent based models assume the conservation of number of agents and wealth. Both these assumptions are unrealistic. In this paper, we study the limiting wealth distribution when one or both of these assumptions are not valid. Given the universality of the law, we have tried to study the wealth distribution from the asset exchange models point of view. We consider models in which a) new agents enter the market at constant rate b) richer agents fragment with higher probability introducing newer agents in the system c) both fragmentation and entry of new agents is taking place. While models a) and c) do not conserve total wealth or number of agents, model b) conserves total wealth. All these models lead to a power-law tail in the wealth distribution pointing to the possibility that more generalized asset exchange models could help us to explain emergence of power-law tail in wealth distribution.

1 Introduction

A century ago, an Italian social economist Pareto collected and studied data of distribution of income across several European countries. He observed that 80% of the income is in 20% hands and the distribution of income has a power-law tail, *i.e.* $p(x) \propto x^{-1-\nu}$, where $p(x)$ probability that an individual has income x . The exponent ν is called Pareto index. The exponent measured by him for different kingdoms and countries varied between 1.1 to 1.7. The distribution of wealth also shows a similar behavior. The validity of Pareto law was questioned and reexamined many times. In modern times, the Japanese, Australian and Italian personal income distribution have been shown to demonstrate a log normal distribution for lower income coupled with power-law tail [1, 2, 3]. For wealth distribution, the distribution of wealth in rich indian families has a power-law tail [4]. Same feature is observed in the wealths of Hungarian aristocratic families [5]. Even for ancient Egyptian society, it has been conjectured that the wealth distribution had power-law tail [6]. The empirical studies on

¹e-mail contact :ali@cms.unipune.ernet.in

²e-mail contact : gade@unipune.ernet.in

data of the distribution of income and wealth in modern USA and UK show a power-law tail as well[7]. All these studies suggest the existence of power-law tail of wealth distribution and income distribution in different societies in different parts of the world and it seems to be true in older as well as modern times. However, the value of the exponent changes in different societies. We will be attempt to explain wealth distribution from the viewpoint of asset exchange models in this paper.

Given that the universality of Pareto law is so robust, cutting across economies which follow different financial system, and even across time, there must be a simple explanation for this feature. Several models have been proposed from this viewpoint. There are attempts to use ideal-gas like models which recover these features [8]. Models in analogy with directed polymers in random media [9] have been proposed. Generalized Lotka-Volterra dynamics [10] and stochastic evolution equation which incorporate trading as well as random changes in prices of investments [11] have also been proposed.

Trading is an economic activity which is common to all systems in all countries and has been so from time immemorial. Thus asset exchange models which are simplest models of economic transaction should give us an explanation of Pareto law. There have been several attempts in this direction [12, 13, 14, 15, 16]. We try to study it from the viewpoint of asset exchange models. In literature, two types of asset exchange models have been studied extensively. In these models, there is neither consumption nor production of wealth. One of models is called yard-sale model (YS) and other is known as theft-fraud (TF) Model [17]. In YS model, the amount at stake is certain fraction of wealth of poorer agent while in TF model, both agents put a certain fraction of their wealth to stake. However, none of these models reproduces the power-law distribution of wealth. The YS model leads to condensation of wealth in the hand of one agent asymptotically while the TF model which is an ideal-gas like model gives us an exponential distribution of wealth. In this context, several approaches have been attempted to reproduce power-law tail in wealth distribution starting from asset exchange models. Sinha showed that modified YS rule in which poorer player wins with higher probability leads to a power-law distribution of wealth [18]. In a previous paper, we showed that mixing the above two models leads to power-law tail in resultant wealth distribution [15]. Several other variants like introducing altruism in YS and TF models [19], introducing saving propensity in TF type models [13, 14] etc. have been studied.

A common and rather unrealistic feature of these models is that the rules do not allow total wealth in a society to fluctuate, nor the number of players in the society change in time. It is clear that change in working population makes an impact on the wealth distribution of the country. It is also clear that the true GDP (Gross Domestic Product) of the world has increased over time. Thus the total wealth is not conserved. In fact, projection of real GDP growth is obtained by summing the estimates of the percentage changes in: increased labor inputs, increased capital inputs and productivity growth [20]. Thus it is clear that increase in labor supply will increase the real GDP (the proportionality

constant is elasticity of output with respect to labor). In most societies, this number keeps growing. (Of course, there are exceptions to this rule of thumb. In older times, catastrophic events like drought, earthquake, plague have wiped out populations. In modern economies, even decrease in labor supply is possible. The populations in countries like Russia, Italy, Ukraine etc. are reducing and economists are discussing its consequences [21]. However, this is a new trend and its outcome will be apparent only after a decade or so [22].) Labor supply depends on demography. Kitov in has postulated that the GDP growth rate depends on relative change in the number of people with a specific age (9 years in the USA) [23]. Thus the linear relationship between growth of real GDP and growth of (working) population is a reasonable assumption. We are not taking into account the impact of decreasing labor participation ratio on economy in this paper which is a new phenomenon. We are not taking into account the impact of productivity or fitness of the agents or influx of new capital by agents already in the system. We will deal with the only effect that the total wealth also keeps increasing since increasing population discovers newer sources of income. We will systematically analyze the impact of increase in number of agents in the Yardsale model of asset exchange.

The wealth often splits at the higher end of the spectrum. Rich people have descendents who become independent agents in their own right. Similarly, large corporations split into entities which function independently. There is a fragmentation of wealth when people have children or companies split. Apart from the fact that large wealth is difficult to manage, there are social and legal pressures which encourage division of wealth of rich people. The society at large, resents concentration of wealth in hands of some people leading to income inequalities. The government, on the other hand, wants to discourage monopolies from the perspective of encouraging economic efficiency and puts in measures like antitrust act, ceiling act, quotas [24, 25] etc. These measures affect rich people more than the poor.

There are previous attempts to take into account these factors. Slanina has given a model with nonconserved wealth but conserved number of agents. He models the wealth distribution in analogy with inelastically scattering particles and reports a power-law distribution with Pareto index in the interval [1,2] depending on a free parameter introduced in the model [26]. There is an attempt to take into account splitting of wealth between agents. R. Coelho et al introduced the family-network model for wealth distribution in societies. Here, they assume fragmentation of wealth of older agents among their neighbors. This agent reappears with zero age and gets linked to two randomly selected agents that have wealth greater than a minimal value q . The wealth q is taken away from the wealth of that selected agents and it is redistributed in a random and preferential manner in society. This model leads to Pareto-like power-law tail for the upper 5% of the society. The Pareto index in this model found to be 1.8 [27]. But this model is static, and total wealth and number of agents are conserved. Lee and Kim introduced the model with nonconserved number of agents similar to the model of growing network. In that model, the number of agents increases linearly with the time but the model does not consider any

exchange of assets, *i. e.* flow of money between the agents. The wealth production for any agent is due to intrinsic ability to produce wealth. This model leads to power-law tail of the wealth distribution [28]. As we argued, given the universality of the law, we feel that one should be able to obtain it within the paradigm of asset exchange models. Despite its faults, we believe that YS model is a good model of financial transactions. We make an attempt to explore the ‘design space’ of asset exchange models, in particular that of YS model, by introducing changes in capital and labor. Taking YS model as a basic model, we investigate the asymptotic distribution of wealth with nonconserved of number of agents and/or the total wealth.

We introduce three different models in this context. We investigate models mimicking introduction of newer wealth (with newer agents), fragmentation of wealth with newer agents coming into play and a model in which both the processes are occurring.

2 The Model(s) and Simulations

We focus on YS model of asset exchange. In YS model, which we believe to be basically correct description of the asset exchange, the rule is the following: the wealth exchanged between two players is fraction of the wealth of the poorer player. Mathematically, we define it as: if we have agents i and j have wealth $x_i(t)$ and $x_j(t)$ at time t and are chosen to be updated. Their wealth at next time step will be:

$$x_i(t+1) = x_i(t) + \Delta x \quad (1)$$

$$x_j(t+1) = x_j(t) - \Delta x \quad (2)$$

$\Delta x = \alpha \min(x_i(t), x_j(t))$ Where α random number in the interval $[0,1]$. The wealth of all other agents does not change $x_k(t+1) = x_k(t)$ for all k different from i and j .

As mentioned before, this model leads to the unrealistic outcome that the entire wealth is owned by one agent asymptotically. As mentioned in the previous section, various modifications of this model do not yield a satisfactory power-law tail either. We would like to make a change which we believe is realistic and has not been taken into account before.

There are two processes which need to be accounted for:

a) The economic system is not closed and newer agents come in bringing in their own money. As we stated before it is reasonable to assume that the true GDP growth will be proportional to labor supply or number of agents.

b) The wealth often splits at the higher end of the spectrum. Rich people have progeny who become independent agents in their own right. Similarly, large organizations often split into daughter organizations for smoother management. We explore variants of YS model in which richer agents keep splitting with higher probability. In these models the number of agents is not conserved but increase in time (which is very realistic) and see an impact of such a system in the wealth

distribution pattern. We believe YS process where the asymptotic state is a condensate, naturally leads to merging. We explicitly introduce fragmentation proportional to the wealth of an agent. We also probe third situation in which both the events of newer agents entering the market and division of wealth for some agents keep happening.

We will study both the effects individually and also study the wealth distribution when both of the above effects are introduced.

We start with a pool of $N_0 = 100$ players that have wealth selected randomly from uniform distribution in the interval $[0,1]$ in all models below. We fix the value of $\alpha = 0.5$ in all cases. (We checked that changing value of α to another constant does not change results. Changing α randomly in time, or assigning a quenched random value of α to each agent does not change the steady state, if any.) Probability density of wealth is approximated from a histogram with very fine bins. We have used 10^5 bins of uniform size in all models below except model B where we used 10^4 bins. After the simulation, we compute the relative wealth of each agent, normalized by total wealth in the system. (Due to scale invariance of power-laws, this linear transformation does not change the nature or exponent of the power-law tail.) We compute probability histogram of the relative wealth and normalize it by total number of observations as well as by the bin-size to obtain an estimator for probability density function of the wealth [29]. This procedure is followed in all the case below. This distribution is noted as $P(\bar{x})$.

(A) Inflow of newer agents: In this model, we have a steady flow of newer agents entering the fray with some wealth. Thus, neither the number of agents nor the total wealth is conserved. Let us denote the number of agents after k rounds of transactions by N_k . After each round of transactions, the number increases by one and thus $N_k = N_0 + k$. Each round of transactions consists of as many transactions as the number of agents, giving each agent a chance to have a couple of transactions on an average. After every round, a new agent enters the system, that agent has wealth selected randomly from uniform distribution in the interval $[0,1]$. Transactions take place according to YS rule by choosing two agents randomly to make trade. We will demonstrate that this system indeed reaches a steady state with stable asymptotic probability distribution of wealth.

We start the simulation with $N_0 = 100$. We find the probability distribution of wealth after T time-steps and observe that it indeed converges asymptotically. We average over 100 initial configurations. Total wealth increases linearly with the number of agents. However, we find that the wealth of the richest agent occupies a fraction of total wealth which remains constant asymptotically. Thus condensation is clearly not reached. In order to compare the distributions at different times, we obtain the probability distribution of \bar{x} where \bar{x} is fraction of wealth that a given agent has acquired. (Since the power-laws are scale invariant, this transformation does not change the power-law nature of tail.) We show the probability distribution $p(\bar{x})$ after $T = 10^4$ and $T = 10^5$ time-steps in Fig. 1. The distribution has clearly converged. We observe that this wealth distribution has a power-law tail with exponent 1.5(5). Pareto index in this model $\nu = 0.5(5)$. This curve has lot of fluctuations. To average them out

we define a new variable $f(\bar{x}) = \int_{\bar{x}}^{\infty} p(\bar{x})d\bar{x}$ which is proportional to probability of having wealth greater than \bar{x} . Clearly $f(\bar{x}) \propto (\bar{x})^{-\nu}$ and displays a power-law tail if $p(\bar{x}) \propto (\bar{x})^{-\nu-1}$ for larger \bar{x} . In the inset, we plotted $f(\bar{x})$ as function of \bar{x} for $T = 10^4$ and $T = 10^5$ time steps as in original figure. The figure in the inset demonstrates that the probability distribution has indeed converged to a steady state.

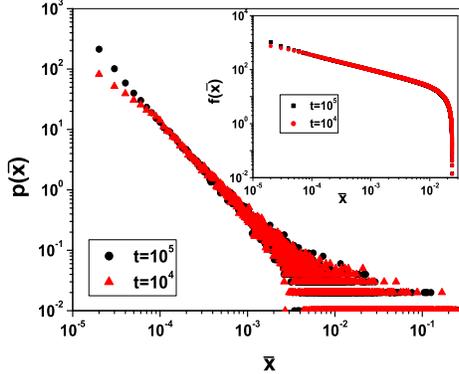


Figure 1: We plot wealth distribution function $p(\bar{x})$ as a function of \bar{x} for model A for $T = 10^4$ and $T = 10^5$ timesteps. We clearly see that the model has a steady state which has a power-law tail with exponent 1.5(5). Inset: We plot $f(\bar{x}) = \int_{\bar{x}}^{\infty} p(\bar{x})d\bar{x}$ for the same parameters as the original figure. This clearly demonstrates that the system has reached a steady state.

(B) Fragmentation of wealth: We consider a situation in which there is no inflow of wealth and the richer agents fragment. This is a model of closed society where no extra wealth comes in, but population increases. Now inheritances can play a role in which unused material possessions and assets are divided among siblings. However, for poorer agents are unlikely to have unused assets and material possessions which need savings that are translated into investment. It is known that savings rate for the rich are higher than the savings rate for the poor. One more reason why the wealth is likely to be fragmented at higher end is managerial and economic efficiency. We also note that there are social and legal pressures to stop some agent from grabbing all the wealth. Countries have antitrust law and its equivalent to stop monopolies since monopolies not only lead to higher income inequality but they also lead to economic inefficiency. There are land ceiling laws in several countries which essentially encourage agents to fragment their assets when they are too rich. We incorporate this in our model by saying that rich people divide their wealth with higher probability than the poor. After every τ time steps, we pick an agent randomly and with a probability that is proportional to his wealth, we introduce two agents each with half wealth and remove this agent from the pool. (We have checked that change in

the value of τ does not change the asymptotic distribution of wealth.) In other words, if we choose an agent (say k th) randomly, we split his wealth with probability $x_k(t)$. We reduce wealth of k 'th agent to half and introduce the $(N+1)$ th agent with same wealth (we checked that unequal partitions do not affect our results.) In this model, the number of agents keeps increasing while total wealth is conserved. Thus, the average wealth reduces. The probability of fragmentation of wealth of agent is related to absolute value of his wealth. However, richer agents keep getting disproportionately targeted for fragmentation. Thus the entire distribution slowly tends to a delta function at $x = 0$ asymptotically with a power law tail which has decreasing weight as time grows. However, we may look for a quasistationary distribution similar to Slanina model [26] in which there is no conserved average wealth but a quasistationary state with a power law tail. The tail always shows a power-law behavior with exponent $2.8(0)$.

We carry out simulation with $N_0 = 100$ agents for M timesteps and average over 3000 initial conditions. After every $\tau = 10$ timesteps, we attempt a fragmentation of randomly chosen agent as mentioned above. The distribution clearly displays a clear power-law tail at any time. For consistency, we have displayed distribution of normalized wealth \bar{x} though the total wealth remains constant. (The probability distribution of x will not be any different except a scale factor.) We show the wealth distribution after $10^4, 10^5$ and 10^6 time-steps in Fig. 2. It is clear that the distribution has a power-law tail with the exponent $2.8(0)$. *This exponent does not change in time.* We will support this conclusion by a scaling argument.

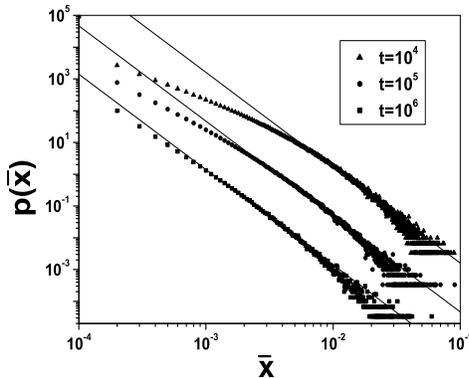


Figure 2: We plotted wealth distribution $p(\bar{x})$ as function of \bar{x} for model B for $10^4, 10^5$ and 10^6 timesteps. We get power-law tail with exponent $2.8(0)$.

The presence of quasistationary state can be demonstrated by the fact that the distribution admits an interesting scaling behavior. In Fig. 3, we plot the $p(\bar{x}, t)\bar{x}^{-\alpha}$ as a function of $\bar{x}^\alpha t$ for different times. The functional forms

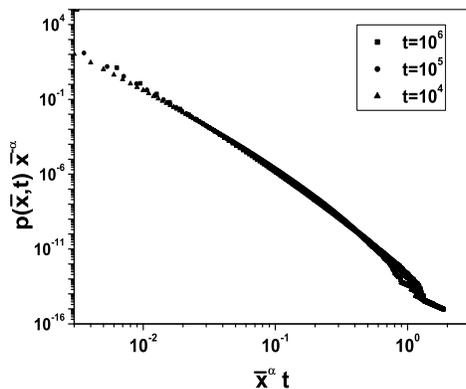


Figure 3: The scaling plots of the wealth distribution obtained from numerical simulation of model B. The data for three different times collapse onto a single curve.

at different times collapse to single curve and the scaling function is $p(\bar{x}, t) \sim \bar{x}^{-\alpha} f(\bar{x}^\alpha t)$ where $\alpha = 2 + \nu$ and ν the Pareto index in this model.

The exponent 2.8(0) is certainly more realistic and comparable with exponents obtained in empirical studies. The Pareto index is $\nu = 1.8(0)$ in this model.

(C) Fragmentation and inflow of agents: This is a model where both the processes described in model A as well as B happen. The society is open. Newer agents keep coming in with new sources of wealth and richer agents keep getting fragmented. Here, neither total wealth nor number of agents is conserved. There are several ways in which this could be achieved. We have tried three different cases:

Case I) In this case, we couple fragmentation of any agent with addition of new agent with random wealth chosen from uniform distribution over $[0, 1]$. We do this to keep average wealth constant asymptotically. Thus every time an agent fragments, we have two new agents (one each due to fragmentation and addition) and on an average wealth of value $1/2$ is introduced in the system when two new agents are created. (This is due to the fact that in our models, any agent has initial wealth chosen from uniform random distribution over $[0, 1]$. We do not give any extra wealth to the agent created due to fragmentation.) Thus average wealth which is the first moment of distribution of wealth will be reaching a constant $1/4$ asymptotically. We have checked that changing α and making fractions unequal do not change results. We start simulation this model with $N_0 = 100$ agents. After carrying out as many YS transactions as the number of agents in the system, we randomly select an agent as the model B. We fragment it with probability proportional to his wealth. Now, we add the new agent with wealth chosen randomly from interval $[0, 1]$ if and only if the

fragmentation has occurred. We averaged over 100 initial conditions. In Fig. 4, we have shown the probability distribution of fractional values of wealth $p(\bar{x})$ after $T = 10^4$ and $T = 10^5$ timesteps. It is clear that system indeed reaches a stationary state with power-law tail. The power-law displayed has an exponent 1.7(7). Pareto index in this model $\nu = 0.7(7)$. As the model A, we plotted $f(\bar{x})$ as function of \bar{x} for $T = 10^4$ and $T = 10^5$ time steps as in original figure. The figure in the inset demonstrates that the probability distribution has indeed converged to a steady state.

Case II) Here we add a new agent (with probability one) after every round of transactions, *i.e.* after as many transactions as the number of agents in the system. This is very much like Model A. Additionally, we also choose an agent randomly after every round of transactions and fragment it with probability proportional to his wealth. In this case, the new agent with his own wealth enters the system at least as frequently as the event of fragmentation of the older agent (which has very little probability). Thus we expect a steady state where the total wealth increases linearly with number of agents. Again, in order to be able to compare distributions of wealth at different times, we obtain the probability distribution $p(\bar{x})$ where \bar{x} is normalized value of wealth. The figures clearly demonstrate a steady state. We have checked also that changing α and making fractions unequal do not change results. We start simulation with $N_0 = 100$ and average over 100 configurations. In Fig. 5, we demonstrate the probability distribution after $T = 10^4$ and after $T = 10^5$ timesteps. This model clearly leads to a distribution with power-law tail with exponent 1.9(0). Pareto index in this case $\nu = 0.9(0)$. As the model A, we plotted $f(\bar{x})$ as function of \bar{x} for $T = 10^4$ and $T = 10^5$ time steps as in original figure as an additional evidence for the approach to a steady state.

Case III) We have also tried another possibility where no particular measures are adopted to make average asymptotic wealth constant. It is unrealistic to assume that new agent enters the fray only after exactly N transactions are completed. However, it is reasonable to assume that number of transactions would be proportional to number of agents. Hence, we make a probabilistic rule that new agent joins with a probability inversely proportional to number of agents in the system at that time. Thus we allow one agent to join with probability $1/N$ after every τ timesteps where N is number of agents at that time. We also attempt to fragment a randomly agent chosen with probability proportional to his wealth after every τ timesteps. Though total wealth keeps increasing with number of new agents joining, it is not necessary that the increase will be linearly proportional to total number of agents. However, our simulations indicate that total number of agents and total wealth keep increasing linearly with time. Thus fractional wealth \bar{x} is a good variable to analyze. We start with $N_0 = 100$ agents. After every $\tau = 10$ timesteps we choose an agent randomly and fragment it with probability proportional to his wealth while a new agent joins with a probability inversely proportional to total number of agents in the system. (We have checked that changing τ does not alter the results.) Here the system does not reach saturation during our simulation time. However, we observe that the distribution has a power-law tail. We have

plotted the wealth distribution $p(\bar{x})$ after $T = 10^6$ timesteps. It clearly has a power-law tail with exponent 2.2(2) (See Fig. 6). After the power-law, there is a small peak in the distribution at very large masses as in case of some particle aggregation models [30]. We average over 100 configuration. Pareto index in this model $\nu = 1.2(2)$ at $T = 10^6$ time steps. Though the steady state is not very clear in this situation, the exponent does not change much at later times. The exponent is high and is comparable to realistic societies.

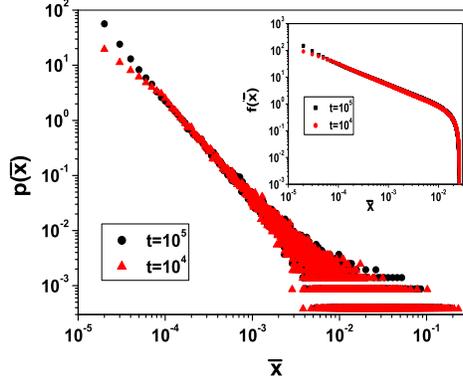


Figure 4: We plot wealth distribution $p(\bar{x})$ as a function of \bar{x} for model C case I for $T = 10^4$ and $T = 10^5$ timesteps. We clearly see that the model has a steady state which has a power-law tail with exponent 1.7(7). Inset: We plot $f(\bar{x}) = \int_{\bar{x}}^{\infty} p(\bar{x})d\bar{x}$ for the same parameters as the original figure. This clearly demonstrates that the system has a steady state.

We have checked that the power-law is a better visual fit than lognormal for all the cases discussed above. Besides, we have checked the goodness of fit by finding χ^2/DoF and R^2 for three models by fitting it a power-law functional form and lognormal fit. The values are given in Table 1. It is clear that R^2 values are higher and very close to unity for power-law fit demonstrating that this model is relevant for higher fraction of data. The χ^2/DoF values are lower by orders of magnitude for a power-law fit, which shows that error is far smaller for this fit in all the cases.

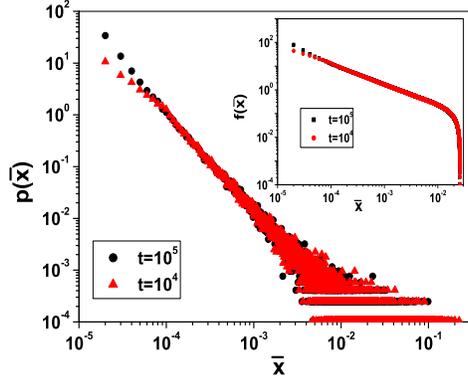


Figure 5: We plot wealth distribution $p(\bar{x})$ as a function of x for model C case II for $T = 10^4$ and $T = 10^5$ timesteps. We clearly see that the model has a steady state which has a power-law tail with exponent 1.9(0). Inset: We plot $f(\bar{x}) = \int_{\bar{x}}^{\infty} p(\bar{x})d\bar{x}$ for the same parameters as the original figure. This clearly demonstrates that the system has a steady state.

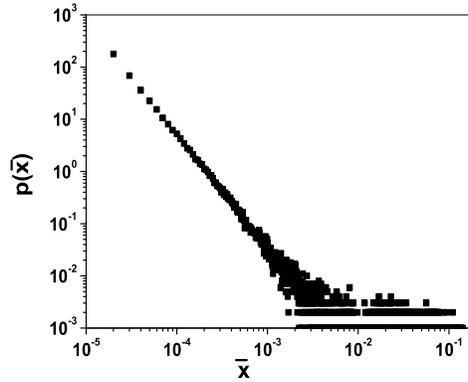


Figure 6: Wealth distribution $p(\bar{x})$ as function of \bar{x} for model C case III for $T = 10^6$ timesteps. We get power-law distribution with exponent 2.2(2) at this time though there is no clear steady state.

Model	Fit	χ^2/Dof	R^2
Model A	Lognormal	5.56×10^2	0.92
	Power-law	1.7×10^{-1}	0.999
Model B	Lognormal	5×10^{-9}	0.04
	Power-law	3.5×10^{-13}	0.999
Model C case I)	Lognormal	1.0×10^5	3.6×10^{-6}
Model C case II)	Power-law	4.8×10^{-1}	0.999
	Lognormal	1.2×10^6	2.8×10^{-7}
Model C case III)	Power-law	1.87	0.999
	Lognormal	2×10^4	0.17
	Power-law	6.0×10^{-2}	0.998

3 Results and discussion

We studied three modifications of the YS model. In all three models, we observe power-law tails in wealth distribution with different exponents. In the first model, we consider an open system, where one new agent decide to join to the system with his wealth after each round of transaction. With this modification of YS, we prevent the condensation of wealth that occurs in the pure YS system and find the power-law wealth distribution with Pareto index $\nu = 0.5(5)$. This index is smaller than ones observed in reality.

In the second model, we prevent the condensation of wealth by allowing the richest agents to fragment into two new agents each. This model was inspired from a similar model trying to give quantitative explanation of power-law tail in the distribution of number of casualties in terrorist attacks observed in empirical data in several countries. This model incorporates fragmentation and merging of terrorist groups. It assumes that, there are a specific number of attack units. (Group of people, weapons, explosives, machines, or even information, which organizes itself to act is a single unit.) Each attack unit has an attack strength. At each time step, one attack unit is chosen with probability which proportional to its strength. This unit chooses to fragment into smaller groups with some probability q and coalesces with another attack unit (again chosen with probability proportional to its strength) with probability $1 - q$. This model exhibits stationary state with power-law distribution for the strength of attack units. The exponent is 2.5 [31], (*i. e.* Pareto index 1.5). We believe that merging occurs naturally in YS model. We introduced the idea of fragmentation proportional to the wealth of an agent in our model B which gave us a power-law wealth distribution with Pareto index $\nu = 1.8(0)$. This value is comparable to ones observed in realistic societies. This model does not have a steady state. But the fact that it admits an interesting scaling behavior, shows the presence of a quasistationary state.

We have also studied a third model which incorporates both ingredients of bringing in newer wealth and fragmentation of richer agents. These are very realistic features of societies. In this model, we always carry out fragmentation

of some randomly chosen agent with probability proportional to its wealth. Here we studied three cases. In the first case, we coupled the fragmentation of any agent with addition of new agent having random wealth. This leads to a distribution with Pareto exponent 0.7(7). In the second case, after every round of transactions, you add an agent and also attempt fragmentation. This leads to Pareto exponent 0.9(0). In the third case, we add an agent with probability inversely proportional to number of agents and also try fragmenting a randomly chosen agent. This leads to distribution with Pareto exponent 1.2(2). Except this case, we have demonstrated that a stationary or quasistationary state is reached asymptotically. We would like to mention that the models are robust with respect to change in parameters τ and α and results do not depend on the precise values of these parameters.

Similar models have been studied in nonequilibrium statistical physics in the context of aggregation models. Takayasu has studied a system in which particles are injected at a steady rate, they diffuse and try to form an aggregate. [32] Asymptotic state here is known to have a power-law tail. There are also models in which particles chip off from aggregates. (For a survey of these different models, see [30]) In our model, the wealth is not discrete, and YS process is similar to but not the same as coagulation of particles. However, some analytic insights could be gained from the analysis of particle aggregation models and studies are being carried in this direction.

MAS thanks Govt. of Yemen for financial support and PMG thanks S. Sinha for discussions.

References

- [1] Y. Fujiwara, W. Souma, H. Aoyama, T. Kaizoji and M. Aoki, *Physica A* **321** (2003) 598.
- [2] H. Aoyama, W. Souma and Y. Fujiwara, *Physica A* **324** (2003) 352.
- [3] F. Clementi and M. Gallegati, *Physica A* **350** (2005) 427.
- [4] S. Sinha, *Physica A* **359** (2006) 555.
- [5] G. Hegyi, Z. Néda and M. A. Santos, *Physica A* **380** (2007) 271.
- [6] A. Y. Abul-Magd, *Phys. Rev. E* **66** (2002) 057104.
- [7] A. Dragulescu and V. M. Yakovenko, *Physica A* **299** (2001) 213.
- [8] A. Dragulescu and V. M. Yakovenko, *Eur. Phys. J. B* **20** (2001) 585 .
- [9] M. Marsili, S. Maslov and Y. -C. Zhang, *Physica A* **253** (1998) 403.
- [10] S. Solomon and P. Richmond, *Physica A* **299** (2001) 188 .
- [11] J. -P. Bouchaud and M. Mezard, *Physica A* **282** (2000) 536 .

- [12] S. Ispolatov, P. L. Krapivsky and S. Render, *Eur. Phys. J. B* **2** (1998) 267.
- [13] A. Chakraborti and B. K. Chakrabarti, *Eur. Phys. J. B* **17** (2000) 167.
- [14] A. Chatterjee, B. K. Chakrabarti and S. S. Manna, *Physica A* **335** (2004) 155 .
- [15] M. A. Saif and P. M. Gade, *Physica A* **384** (2007) 448.
- [16] *Econophysics of Wealth Distribution*, ed. A. Chatterjee, S. Yarlagadda and B. K. Chakrabarti (Springer, Milan, 2005).
- [17] B. Hayes, *American Scientist* **90** (2002) 400.
- [18] S. Sinha, *Phys. Scr. T* **106** (2003) 59.
- [19] M. Rodriguez-Achach and R. Huerta-Quintanilla *Physica A* **361** (2006) 309.
- [20] *Economics: Principles and Policy*, W. J. Baumol and A. S. Blinder (Thomson South-Western, 2005), *Macroeconomics*, A. B. Abel and B. S. Bernanke (Pearson Custom Publishing, 2005) (see also [22].)
- [21] *The Future of Europe: Reform Or Decline*, A. Alesina, F. Giavazzi (MIT Press, 2006).
- [22] In other contexts, the consequences of decreased labor participation have been discussed. See for example ' As Boomers Slow Down, So Might the Economy' in *Regional Economist* <http://www.stls.frb.org/publications/re/2007/c/pages/baby-boomers.html>.
- [23] <http://inflationusa.blogspot.com/2007/10/gdp-growth-rate-and-population.html>.
- [24] *Economics*, J. E. Stiglitz, (W. W. Norton and Company Limited, 1993), *Antitrust*, H. Hovenkamp (West Group, 2005).
- [25] *India's Silent Revolution: The Rise of the Lower Castes in North India*, C. Jaffreot (Columbia University Press, 2003).
- [26] F. Slanina, *Phys. Rev. E* **69** (2004) 046102.
- [27] R. Coelho, Z. Néda, J. J. Ramasco and M. A. Santos, *Physica A* **353** (2005) 515.
- [28] G. Lee and G. Kim, *Physica A* **383** (2007) 677.
- [29] *Probability, Random Variables and Stochastic Processes* 4th ed., A. Papoulis and S. U. Pillai (Tata Mc-Graw Hill 2002).
- [30] S. N. Majumdar, S. Krishnamurthy and M. Barma, *J. Stat. Phys.* **99**, (2000) 1.

- [31] Neil F. Johnson et al, preprint phys/0506213.
- [32] H. Takayasu, Phys. Rev. Lett. **63**, (1989) 2563.