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Fractal differential equations and fractal-time dynamical systems

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Abstract. Differential equations and maps are the most frequently studied examples of dynamical systems and may be considered as continuous and discrete time-evolution processes respectively. The processes in which time evolution takes place on Cantor-like fractal subsets of the real line may be termed as fractal-time dynamical systems. Formulation of these systems requires an appropriate framework. A new calculus called F^α-calculus, is a natural calculus on subsets $F \subset \mathbf{R}$ of dimension α, $0 < \alpha \leq 1$. It involves integral and derivative of order α, called F^α-integral and F^α-derivative respectively. The F^α-integral is suitable for integrating functions with fractal support of dimension α, while the F^α-derivative enables us to differentiate functions like the Cantor staircase. The functions like the Cantor staircase function occur naturally as solutions of F^α-differential equations. Hence the latter can be used to model fractal-time processes or sublinear dynamical systems.

We discuss construction and solutions of some fractal differential equations of the form

$$\mathcal{D}^\alpha_{F,t} x = h(x,t),$$

where h is a vector field and $\mathcal{D}^\alpha_{F,t}$ is a fractal differential operator of order α in time t. We also consider some equations of the form

$$\mathcal{D}^\alpha_{F,t} W(x,t) = L[W(x,t)],$$

where L is an ordinary differential operator in the real variable x, and $(t,x) \in F \times \mathbf{R}^n$ where F is a Cantor-like set of dimension α.

Further, we discuss a method of finding solutions to F^α-differential equations: They can be mapped to ordinary differential equations, and the solutions of the latter can be transformed back to get those of the former. This is illustrated with a couple of examples.

Keywords. Fractal-time dynamical systems; fractal differential equations; fractal calculus; Cantor functions; subdiffusion; fractal-time relaxations.

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1. Introduction

It is now a well-known fact that many structures found in nature can be modelled by fractals [1–3]. The mathematical properties of fractals are also substantially explored [1,4–7].

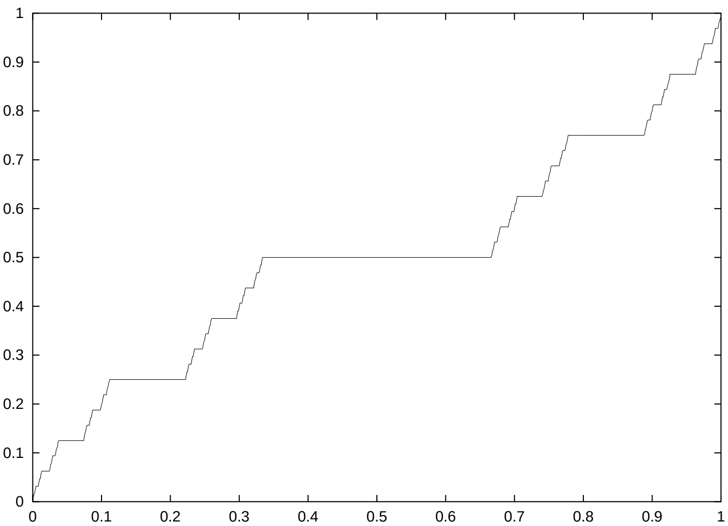

Figure 1. The staircase function $\Gamma(\alpha+1)S_C^\alpha(x)$ for the middle $\frac{1}{3}$ Cantor set.

Fractals are often so irregular that defining smooth, differentiable structures on them is impossible. The methods of ordinary calculus are either inapplicable or ineffective. For example, the ordinary derivative of a Lebesgue–Cantor staircase function (see figure 1 for its graph) is zero almost everywhere in the Lebesgue sense. Consequently this function is not a solution of an ordinary differential equation. This simple example brings out the need to extend ordinary calculus in order to address problems involving fractal structures and phenomena. Fractional calculus [8–11] fills in this gap to some extent. Fractional kinetic equations, fractional master equations etc. [12–18] do provide remarkable models for several phenomena involving scaling and memory effects. But most of the fractional operators are non-local and therefore not suitable to construct causal models involving local scaling.

To circumvent this, fractional derivatives were renormalized to construct local fractional differential operators [19–22], which were further explored in [23,24]. These operators brought out a striking connection between the local fractional differentiability of a function and the Holder exponent/the dimension of its graph.

Another remarkable approach, viz. 'analysis on fractals', is extensively used for the treatment of diffusion, heat conduction, waves, etc., on self-similar fractals [25–28]. Harmonic analysis on fractals using measure theoretical approach [29,30] is a further wonderful development. However, a simple and direct approach is still desired.

In [31], a new calculus (called F^α-calculus) based on fractals $F \subset \mathbf{R}$ is developed with a Riemann-like approach for integrals. It includes formulation of integrals and derivatives of orders $\alpha \in (0,1]$, based on fractals $F \subset \mathbf{R}$. These are called F^α-integrals and F^α-derivatives respectively. This formulation is more intuitive, more direct and transparent, and has a distinct advantage from an algorithmic point of view. The F^α-integral is best suited to integrate functions with α-dimensional fractal supports. In particular, if $F \subset \mathbf{R}$ is a fractal of dimension $\alpha, 0 < \alpha \leq 1$,

Table 1. A few analogies between F^α-calculus and ordinary calculus.

Ordinary calculus	F^α-calculus
$\mathbf{R}$	An α-perfect set F
Limit	F-limit
Continuity	F-continuity
$\int_0^y x^n \, \mathrm{d}x = \frac{1}{n+1} y^{n+1}$	$\int_0^y (S_F^\alpha(x))^n \, \mathrm{d}_F^\alpha x = \frac{1}{n+1} (S_F^\alpha(y))^{n+1}$
$\frac{\mathrm{d}}{\mathrm{d}x} x^n = n\, x^{n-1}$	$\mathcal{D}_F^\alpha((S_F^\alpha(x))^n) = n\,(S_F^\alpha(x))^{n-1}\,\chi_F(x)$
Leibniz rule	F^α-Leibniz rule
Fundamental theorems	Fundamental theorems of F^α-calculus
Integration by parts	Rules for F^α-integration by parts
Taylor expansion	'Fractal' Taylor expansion

then the F^α-integral $\int_0^x \chi_F(y)\, \mathrm{d}_F^\alpha y$ of the characteristic function χ_F of F is the integral staircase function $S_F^\alpha(x)$ which is a generalization of the Cantor staircase function. The F^α-derivative is best suited for functions like the Cantor staircase which 'change' only on a fractal. As expected, the F^α-derivative of $S_F^\alpha(x)$ is $\chi_F(x)$. In fact, staircase functions play a central role in F^α-calculus: a role analogous to that of the independent variable itself in ordinary calculus. In the definitions of F^α-integral and F^α-derivative, the quantity $(S_F^\alpha(y) - S_F^\alpha(x))$ replaces $(y - x)$, the length of the interval $[x, y]$, or the distance between x and y.

Several results in the F^α-calculus are analogous to standard results of classical calculus such as product rule, fundamental theorems, etc. (see table 1). This is what makes it more intuitive and transparent.

The F^α-differential equations are differential equations involving the F^α-derivatives exactly like the ordinary differential equations involving the ordinary derivatives. Since staircase-like functions occur naturally as their solutions, F^α-differential equations offer possibilities of modeling dynamical behaviours naturally for which ordinary differential equations and methods of ordinary calculus are inadequate. In this paper we consider some examples for the purpose of illustration.

Under suitable conditions, it is possible [32] to map the F^α-integrals and F^α-derivatives to ordinary integrals and derivatives of appropriately defined functions. Indeed, as discussed in §4 with examples, and in [32], it is possible to solve certain fractal differential equations by mapping them to ordinary differential equations and fractalizing the solutions back.

The organization of the paper is as follows: In §2, we present a review of F^α-calculus. It includes defining mass function and staircase function, F^α-integration, and F^α-derivative. A few of the essential properties proved in [31] are stated.

Sections 3 and 4 deal with the main theme of the paper. These sections are written in more intuitive fashion avoiding the jargon as far as possible. In §3, we discuss some examples of F^α-differential equations. The conjugacy between F^α-calculus and ordinary calculus developed in [32] is sketched in §4 and further details are given in Appendix A. The use of this conjugacy in solving F^α-differential equations is illustrated in §5. Section 6 is the concluding section.

We begin by reviewing the F^α-calculus.

2. Review of calculus on fractals

2.1 *Staircase functions*

In this section we take a brief review of calculus on fractal subsets of real line, or F^α-calculus, which is developed in [31]. It involves the definitions of mass function, staircase function, F^α-integration and F^α-differentiation. Here we just state definitions and theorems for ready reference. Detailed proofs can be found in [31].

DEFINITION 1

A subdivision $P_{[a,b]}$, or just P, of the interval $[a,b]$, $a<b$, is a finite set of points $\{a=x_0,x_1,\ldots,x_n=b\}$, $x_i<x_{i+1}$. Any interval of the form $[x_i,x_{i+1}]$ is called a component interval or just component of the subdivision P.

DEFINITION 2

For a set $F\subset\mathbf{R}$ and a subdivision $P_{[a,b]}$, $a<b$, the mass function $\gamma^\alpha(F,a,b)$ is given by

$$\gamma^\alpha(F,a,b)=\lim_{\delta\to 0}\inf_{\{P_{[a,b]}:|P|\le\delta\}}\sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^\alpha}{\Gamma(\alpha+1)}\theta(F,[x_i,x_{i+1}]), \qquad (1)$$

where $\theta(F,[x_i,x_{i+1}])=1$ if $F\cap[x_i,x_{i+1}]$ is non-empty, and zero otherwise, and

$$|P|=\max_{0\le i\le n-1}(x_{i+1}-x_i),$$

the infimum being taken over all subdivisions P of $[a,b]$ such that $|P|\le\delta$.

The motivations for this definition come from fractional calculus and the construction of Hausdorff measure.

Among several nice properties of the mass function [31], we particularly note the interval-wise additivity and behaviour under scaling and translation:

1. (Interval-wise additivity). Let $a<b<c$ and $\gamma^\alpha(F,a,c)<\infty$. Then $\gamma^\alpha(F,a,c)=\gamma^\alpha(F,a,b)+\gamma^\alpha(F,b,c)$.
2. (Translation). For $F\subset\mathbf{R}$ and $\lambda\in\mathbf{R}$, let $F+\lambda$ denote the set $F+\lambda=\{x+\lambda : x\in F\}$. Then, $\gamma^\alpha(F+\lambda,a+\lambda,b+\lambda)=\gamma^\alpha(F,a,b)$.
3. (Scaling). For $F\subset\mathbf{R}$ and $\lambda\ge 0$, let λF denote the set $\lambda F=\{\lambda x : x\in F\}$. Then $\gamma^\alpha(\lambda F,\lambda a,\lambda b)=\lambda^\alpha\,\gamma^\alpha(F,a,b)$.

DEFINITION 3

Let a_0 be an arbitrary but fixed real number. The integral staircase function $S_F^\alpha(x)$ of order α for a set F is given by

$$S_F^\alpha(x)=\begin{cases}\gamma^\alpha(F,a_0,x) & \text{if } x\ge a_0\\ -\gamma^\alpha(F,x,a_0) & \text{otherwise.}\end{cases} \qquad (2)$$

The number a_0 can be chosen according to convenience. If $\gamma^\alpha(F,a,b)$ is finite for a set $F\subset\mathbf{R}$ and some $\alpha\in(0,1]$, the important properties relating the staircase function are:

1. $S_F^\alpha(x)$ is increasing in x.
2. If $F \cap (x,y) = \emptyset$, then S_F^α is a constant in $[x,y]$.
3. $S_F^\alpha(y) - S_F^\alpha(x) = \gamma^\alpha(F,x,y)$.
4. S_F^α is continuous on (a,b).

As an example, a staircase function corresponding to $F = C$ for the middle $\frac{1}{3}$ Cantor set is shown in figure 1. This is the same as the Cantor staircase function apart from a multiplying factor.

Incidentally the mass function allows us to formulate a new definition of dimension called γ-dimension [31]. It is given by

$$\begin{aligned}\dim_\gamma(F \cap [a,b]) &= \inf\{\alpha\colon\ \gamma^\alpha(F,a,b) = 0\}\\ &= \sup\{\alpha\colon\ \gamma^\alpha(F,a,b) = \infty\}.\end{aligned}$$

It turns out [31] that $\dim_{\mathcal{H}}(F \cap [a,b]) \le \dim_\gamma(F \cap [a,b]) \le \dim_B(F \cap [a,b])$ where $\dim_{\mathcal{H}}$ and $\dim_B$ denote the Hausdorff dimension and the box dimension respectively. Also,

$$\gamma^\alpha(F,a,b) \ge \frac{1}{\Gamma(\alpha+1)}\mathcal{H}^\alpha(F \cap [a,b]),$$

where $\mathcal{H}^\alpha$ denotes the Hausdorff measure of order α. Further, this relation becomes an equality of F is compact. The details are discussed in [31].

2.2 *α-Perfect sets*

The α-perfect sets (Definition 5) are sets having properties necessary to define F^α-derivative, and which are required also for the formulation of fundamental theorems of F^α-calculus. The correspondence between sets and their staircase functions is many to one. An α-perfect set is basically the representative of a class of sets giving rise to the same staircase function and has certain desired properties.

DEFINITION 4

We say that a point x is a point of change of a function f, if f is not constant over any open interval (c,d) containing x. The set of all points of change of f is called the set of change of f and is denoted by Sch$f$.

DEFINITION 5

Let $F \subset \mathbf{R}$ be such that $S_F^\alpha(x)$ is finite for all $x \in \mathbf{R}$ for some $\alpha \in (0,1]$. Then the set $\mathrm{Sch}(S_F^\alpha)$ is said to be α-perfect.

Some facts regarding an α-perfect set $H = \mathrm{Sch}(S_F^\alpha)$ are listed below.

1. $S_H^\alpha = S_F^\alpha$.
2. If $x \in H$ and $y < x < z$, then either $S_H^\alpha(y) < S_H^\alpha(x)$ or $S_H^\alpha(x) < S_H^\alpha(z)$ (or both).
3. For any point $x \in H$, there is at the most one more point $y \in H$ such that $S_H^\alpha(x) = S_H^\alpha(y)$.
4. The set H is the intersection of all closed sets giving rise to the same staircase function S_F^α. In other words, H is the minimal closed set amongst them.

2.3 *F-limits and F-continuity*

Now we summarize the notation for limits and continuity using the topology of F with the metric inherited from $\mathbf{R}$:

Let $F \subset \mathbf{R}, f\colon\ \mathbf{R} \to \mathbf{R}$ and $x \in F$. A number ℓ is said to be the limit of f through the points of F, or simply F-limit as $y \to x$ if given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$y \in F \quad \text{and} \quad |y - x| < \delta \Longrightarrow |f(y) - \ell| < \epsilon.$$

If such a number exists, then it is denoted by

$$\ell = F\text{-}\lim_{y\to x} f(y).$$

The notions of F-continuity and uniform F-continuity are developed on similar lines [31].

2.4 *Review of the Riemann integral on* $\mathbf{R}$

Since much of the calculus developed in [31] follows Riemann integral approach, we begin by recalling the definition of the Riemann integral in ordinary calculus [33]. Let $g\colon\ [a,b] \to \mathbf{R}$ be a bounded function. Let $I \subset [a,b]$ be any closed interval and let

$$M'[g,I] = \sup_{x\in I} g(x)$$

and

$$m'[g,I] = \inf_{x\in I} g(x).$$

Further, for a subdivision $P = \{y_0, \ldots, y_n\}$ of $[a,b]$, the upper and the lower sums are defined respectively as

$$U'[g,P] = \sum_{i=0}^{n-1} M'[g,[y_i,y_{i+1}]](y_{i+1} - y_i)$$

and

$$L'[g,P] = \sum_{i=0}^{n-1} m'[g,[y_i,y_{i+1}]](y_{i+1} - y_i).$$

Now the upper and the lower Riemann integrals are defined respectively as

$$\overline{\int_a^b} g(y)\,\mathrm{d}y = \inf_P U'[g,P]$$

and

$$\underline{\int_a^b} g(y)\,\mathrm{d}y = \sup_P L'[g,P],$$

where the supremum and the infimum are taken over all the subdivisions P of $[a, b]$. The function g is said to be Riemann integrable over $[a, b]$ if the upper and the lower integrals are equal, and in that case the Riemann integral of g over $[a, b]$ is defined to be the common value.

We carry out a similar construction on (fractal) subsets of $\mathbf{R}$.

2.5 F^α-*integration*

As mentioned earlier, the F^α-integral is suitable to integrate functions with fractal support. The ordinary Riemann integral of such functions is usually undefined or zero.

In what follows, the class of functions f: $\mathbf{R} \to \mathbf{R}$ which are bounded on F is denoted by $B(F)$.

Let $f \in B(F)$. Further, let I be a closed interval. We denote two quantities M and m:

$$M[f, F, I] = \begin{cases} \sup_{x \in F \cap I} f(x) & \text{if } F \cap I \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

and similarly

$$m[f, F, I] = \begin{cases} \inf_{x \in F \cap I} f(x) & \text{if } F \cap I \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

DEFINITION 6

Let F be such that S_F^α be finite on $[a, b]$. For $f \in B(F)$, the lower F^α-integral is given by

$$\underline{\int_a^b} f(x)\,\mathrm{d}_F^\alpha x = \sup_{P_{[a,b]}} L^\alpha[f, F, P], \tag{5}$$

where

$$L^\alpha[f, F, P] = \sum_{i=0}^{n-1} m[f, F, [x_i, x_{i+1}]](S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i)) \tag{6}$$

and the upper F^α-integral is given by

$$\overline{\int_a^b} f(x)\,\mathrm{d}_F^\alpha x = \inf_{P_{[a,b]}} U^\alpha[f, F, P], \tag{7}$$

where

$$U^\alpha[f, F, P] = \sum_{i=0}^{n-1} M[f, F, [x_i, x_{i+1}]](S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i)), \tag{8}$$

both the supremum and infimum being taken over all the subdivisions of $[a, b]$.

We emphasize the appearance of intersection $F \cap I$ in the definition of M and m, and also the use of $(S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i))$ as in a Riemann–Stieltjes sum instead of $(x_{i+1} - x_i)$.

DEFINITION 7

If $f \in B(F)$, we say that f is F^α-integrable on $[a, b]$ if

$$\underline{\int_a^b} f(x)\,\mathrm{d}_F^\alpha x = \overline{\int_a^b} f(x)\,\mathrm{d}_F^\alpha x.$$

In that case the F^α-integral of f on $[a, b]$, denoted by

$$\int_a^b f(x)\,\mathrm{d}_F^\alpha x$$

is given by the common value.

A few properties of F^α-integral are listed below:

1. Let $a < b$ and f be an F^α-integrable function on $[a, b]$. Let $c \in (a, b)$. Then, f is F^α-integrable on $[a, c]$ and $[c, b]$. Further,

$$\int_a^b f(x)\,\mathrm{d}_F^\alpha x = \int_a^c f(x)\,\mathrm{d}_F^\alpha x + \int_c^b f(x)\,\mathrm{d}_F^\alpha x. \tag{9}$$

2. F^α-integration is a linear operation.
3. If $\chi_F(x)$ is the characteristic function of $F \subset \mathbf{R}$, then

$$\int_a^b \chi_F(x)\,\mathrm{d}_F^\alpha x = S_F^\alpha(b) - S_F^\alpha(a). \tag{10}$$

This *key result* is analogous to the relation $\int_a^b \mathrm{d}x = b - a$ in ordinary calculus.

2.6 *F^α-derivative*

As mentioned earlier, the F^α-derivative is suitable to differentiate functions which 'change' only on a fractal (more specifically, functions f such that $\mathrm{Sch} f \subset F$), such as the Cantor staircase function. We note that the ordinary derivative of such a function, at points of change of the function, does not exist and is zero almost everywhere.

DEFINITION 8

If F is an α-perfect set then the F^α-derivative of f at x is

$$\mathcal{D}_F^\alpha(f(x)) = \begin{cases} F\text{-}\lim_{y\to x} \frac{f(y)-f(x)}{S_F^\alpha(y)-S_F^\alpha(x)} & \text{if } x \in F \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

if the limit exists.

We list a few results involving the F^α-derivative:

1. The F^α-derivative is a linear operator.
2. The F^α-derivative of a constant function f: $\mathbf{R} \to \mathbf{R}$, $f(x) = k \in \mathbf{R}$ is zero, i.e. $\mathcal{D}_F^\alpha(f) = 0$. (This is to be contrasted with the fact that the classical fractional derivative of a constant is not necessarily zero.)
3. The derivative of the integral staircase itself is the characteristic function χ_F of F:

$$\mathcal{D}_F^\alpha(S_F^\alpha(x)) = \chi_F(x). \tag{12}$$

   This is a *key relation*.
4. (First fundamental theorem of F^α-calculus). Let $F \subset \mathbf{R}$ be an α-perfect set. If $f \in B(F)$ is an F-continuous function on $F \cap [a, b]$, and

$$g(x) = \int_a^x f(y)\, \mathrm{d}_F^\alpha y$$

   for all $x \in [a, b]$, then

$$\mathcal{D}_F^\alpha(g(x)) = f(x)\chi_F(x),$$

   where $\chi_F(x)$ is the characteristic function of the set F.
5. (Second fundamental theorem of F^α-calculus). Let f: $\mathbf{R} \to \mathbf{R}$ be a continuous, F^α-differentiable function such that $\mathrm{Sch}(f)$ is contained in an α-perfect set F and h: $\mathbf{R} \to \mathbf{R}$ be F-continuous, such that

$$h(x)\chi_F(x) = \mathcal{D}_F^\alpha(f(x)).$$

   Then

$$\int_a^b h(x)\, \mathrm{d}_F^\alpha x = f(b) - f(a).$$

6. 'Fractal' Taylor series: Under the conditions of F^α-differentiability and some others [32], a function h has a Taylor series expansion:

$$h(w) = \sum_{n=0}^{\infty} \frac{(S_F^\alpha(w) - S_F^\alpha(x))^n}{n!} (\mathcal{D}_F^\alpha)^n h(x). \tag{13}$$

The F^α-calculus also has analogues of Rolle's theorem, the law of the mean, Leibniz rule and integration by parts [31].

A few analogies between F^α-calculus and ordinary calculus are listed in table 1.

3. Fractal (F^α-) differential equations

The F^α-differential equations are differential equations involving the F^α-derivatives. A few simple examples are considered in [31] and [32]. They offer possibilities of modeling dynamical behaviours for which ordinary differential equations and methods of ordinary calculus are inadequate. We consider some examples for the purpose of illustration here.

1. The equation

$$\mathcal{D}_F^\alpha y(t) = \chi_F(t)$$

has solution $y(t) = S_F^\alpha(t)$ satisfying the initial condition $y(t) = 0$ at $t = 0$, in view of (12).

2. *Linear F^α-differential equation:*

$$\mathcal{D}_{F,t}^\alpha x = \chi_F(t)Ax,$$

where A is an $n \times n$ constant matrix and $x \in \mathbf{R}^n$, has a solution

$$x(t) = \exp(S_F^\alpha(t)A)x_0$$

which is analogous to the solution $x(t) = \exp(tA)x_0$ of the equation $\mathrm{d}x/\mathrm{d}t = Ax$. We notice that a linear F^α-differential equation has solutions with sub-linear behaviour. Thus equations like this are suitable as models in nonlinear dynamics.

3. *Fractal diffusion equation* [21,31]: This equation, which was proposed and discussed in [21,22,31], is of the form

$$\mathcal{D}_{F,t}^\alpha (W(x,t)) = \frac{\chi_F(t)}{2} \frac{\partial^2}{\partial x^2} W(x,t), \tag{14}$$

where the density W is defined as a function of two arguments $(x,t) \in \mathbf{R} \times \mathbf{R}$ and with a slight change of notation $D_{F,t}^\alpha$ denotes the partial F^α-derivative with respect to time t, χ_F being the characteristic function of F. (This equation may be compared with ordinary diffusion equation $\frac{\partial W(x,t)}{\partial t} = D\frac{\partial^2}{\partial x^2}W(x,t)$.) The exact solution is

$$W(x,t) = \frac{1}{(2\pi S_F^\alpha(t))^{1/2}} \exp\left(\frac{-x^2}{2S_F^\alpha(t)}\right), \quad W(x,0) = \delta(x). \tag{15}$$

This can be recognized as a subdiffusive solution, since S_F^α is known to be bounded by kt^α (k a constant) in simple cases including Cantor sets [34]. We further remark that such solutions are new exact solutions of Chapman–Kolmogorov equation with appropriate transition probability [21], and they correspond to causal and Markovian processes [12–18].

4. *Friction in a fractal medium* [31]: We consider one-dimensional motion of a particle undergoing friction. First we recall the equation of motion in a continuous (i.e. non-fractal) medium. If the frictional force is proportional to the velocity, the equation of motion can be written as

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -k(x)v, \tag{16}$$

where $k(x)$, the coefficient of friction, may be dependent on the particle position x. Equation (16) can be casted in the form

$$\frac{\mathrm{d}v}{\mathrm{d}x} = -k(x) \tag{17}$$

which is readily solved by integrating $k(x)$ if $k(x)$, which models the frictional medium, is smooth.

If the underlying medium is fractally distributed (e.g. cloud-like), then (17) is inadequate to model the motion. Instead we propose the F^α-differential equation of the form

$$\mathcal{D}_F^\alpha(v(x)) = -k(x) \tag{18}$$

for this scenario. Here, the set F is the support of $k(x)$ which describes the underlying fractal medium, and α is the γ-dimension of F. The physical dimensions of $k(x)$, the coefficient of friction, are fractional.

The solution of (18) is easily seen to be

$$v(x) = v_0 - \int_{x_0}^{x} k(x')\,\mathrm{d}_F^\alpha x', \tag{19}$$

where v_0 and x_0 are the initial velocity and position respectively. In a simple case where $k(x)$ is uniform on the fractal, i.e. $k(x) = \kappa\chi_F(x)$ where κ is a constant, (19) reduces to

$$v(x) = v_0 - \kappa(S_F^\alpha(x) - S_F^\alpha(x_0)).$$

In the extreme cases we obtain back the classical behaviour: (i) If F is empty (frictionless case), then $v(x) = v_0$; (ii) If $F = \mathbf{R}$ (uniform medium) then $v(x) = v_0 - \kappa(x - x_0)$.

5. *Stretched exponential decays* (see §5): The equation

$$\mathcal{D}_F^\alpha \mu(t) = -k\mu(t)\chi_F(t) \tag{20}$$

has a solution

$$\mu(t) = A\exp(-kS_F^\alpha(t)) \tag{21}$$

with the initial condition $\mu(0) = A$. Details of the method to obtain the solution are discussed in §5.

These are simple models based on F^α-differential equations. The solutions of F^α-differential equations naturally involve staircase-like functions. Staircase functions such as the Lebesgue–Cantor staircase function are known to be bounded [34] by sublinear power laws ($bt^\alpha \le S_F^\alpha(t) \le at^\alpha$). Also, they 'change' or 'evolve' only on a fractal set. Thus, this framework may be useful in modeling many cases of sublinear behaviour, fractal time evolution, fields due to fractal charge distributions, etc.

Continuous-time dynamical systems are associated with ordinary differential equations, and discrete-time dynamical systems are associated with maps/diffeomorphisms. But as realized in [21], the dynamical systems associated with F^α-differential equations are a new class of dynamical systems and correspond to those evolving on fractal subsets of time-axis.

4. Fractalizing and defractalizing

We begin the discussion of the conjugacy between the F^α-calculus and the ordinary calculus by considering the following simple example which helps to expose the basic idea.

Let $J = [0,1]$ and $C =$ middle $\frac{1}{3}$ Cantor set. Then one can associate with every point $x \in J$ a point of C and vice versa by the map $f\colon J \to C$ which is defined as follows. Let us represent $x \in J$ by its binary representation, say $x = 0.x_1x_2\ldots$. Thus each $x_i = 0$ or 1. On the other hand, points of C are represented in a base-3 representation. But since the points of C can be represented by using the digits $\{0,2\}$ only (the digit 1 does not appear since middle third parts are always omitted), any point of C has the form $0.y_1y_2\ldots$ where $y_i = 0$ or 2. Let $g : \{0,1\} \to \{0,2\}$ so that $g(0) = 0$ and $g(1) = 2$. Then f is given by $f(x) = f(0.x_1x_2\ldots) = y = 0.y_1y_2\ldots$ where $y_i = g(x_i)$.

The inverse map f^{-1} is obtained by setting $f^{-1}(y) = x = 0.x_1x_2\ldots$ where $x_i = g^{-1}(y_i)$.

The map $f\colon J \to C$ may be called a fractalizing map and $f^{-1}\colon C \to J$ a defractalizing map.

Associated with such fractalizing transformations are the naturally induced maps which take Riemann integrals in ordinary calculus to F^α-integrals and ordinary derivatives to F^α-derivatives. Conversely, maps which take F^α-integrals to Riemann integrals and F^α-derivatives to ordinary derivatives are associated with defractalizing transformations.

Indeed these ideas can be formulated more generally and are developed in [32]. They allow us to establish a conjugacy of ordinary calculus and F^α-calculus. A brief outline is given in Appendix A for ready reference. Here we only sketch the relavent procedure.

Let f be an F^α-integrable function over $[a,b]$. Denote

$$f_1(x) = \int_a^x f(x')\, \mathrm{d}_F^\alpha x', \quad x \in [a,b].$$

By the defractalization process sketched above (and detailed in Appendix A), we obtain a function g, also denoted by $g = \psi[f]$, which is Riemann integrable over $[S_F^\alpha(a), S_F^\alpha(b)]$ (see Theorem 15 in Appendix A). Further, denoting

$$g_1(y) = \int_{S_F^\alpha(a)}^y g(y')\, \mathrm{d}y', \quad y \in [S_F^\alpha(a), S_F^\alpha(b)],$$

we are assured (again by Theorem 15 in Appendix A) that

$$g_1(S_F^\alpha(x)) = f_1(x), \quad x \in [a,b].$$

If we denote the indefinite F^α-integration operator by I_F^α and the indefinite Riemann integration operator by I, then this can be summarized by the relation

$$I_F^\alpha = \phi^{-1} I \psi, \tag{22}$$

where ϕ is just a restriction of ψ to a smaller class of functions (see Appendix A). It can be expressed by a commutative diagram as shown in figure 2.

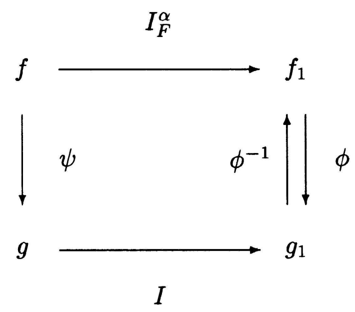

Figure 2. The relation between F^α-integral and Riemann integral.

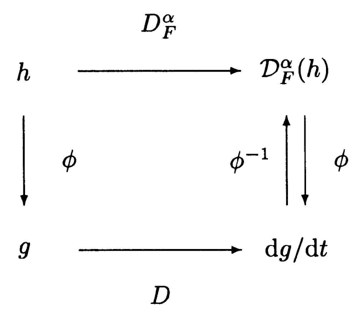

Figure 3. The relation between F^α-derivative and ordinary derivative.

Similar procedure is associated with F^α-derivatives. Under certain conditions (see Appendix A and [32]), if a function h is F^α-differentiable at x, then the function $g = \phi[h]$, obtained by defractalization, is differentiable at $y = S_F^\alpha(x)$ in the ordinary sense. Further, we are assured [32] that

$$\left.\frac{\mathrm{d}g}{\mathrm{d}y}\right|_{y=S_F^\alpha(x)} = \mathcal{D}_F^\alpha(h(x)).$$

If we denote the F^α-differentiation operator by $\mathcal{D}_F^\alpha$ and the ordinary derivative operator by D, then this relation can be summarized as

$$D_F^\alpha = \phi^{-1} D \phi. \tag{23}$$

It can also be expressed by a commutative diagram as shown in figure 3.

As a concrete example of using this conjugacy to calculate F^α-integrals, we calculate the F^α-integral of the function $f(x) = \exp(-x)\,\chi_C(x)$ numerically, where C is the middle $\frac{1}{3}$ Cantor set and $\alpha = \ln(2)/\ln(3)$ is its γ-dimension. Figure 4 shows the results of the numerical calculation. Figure 4f shows the numerical difference $(I_F^\alpha - \phi^{-1} I \psi) f(x)$ which goes to zero as we use finer and finer subdivisions.

More interestingly these maps further provide us with a method for constructing solutions of F^α-differential equations from known solutions of ordinary differential equations. We now proceed to discuss such constructions.

5. Solving F^α-differential equations using the conjugacy

In this section we illustrate solving F^α-differential equations using the conjugacy, with the help of a couple of examples.

To illustrate the procedure, we begin with a simple F^α-differential equation

$$\mathcal{D}_F^\alpha y(x) = \chi_F(x) \tag{24}$$

with the initial condition $y(0) = 0$. Let $g = \phi[y]$ denote the defractalization of y. It is immediate from the definition of map ϕ (Appendix A) that $\phi[\chi_F] = 1$, the constant function. Further,

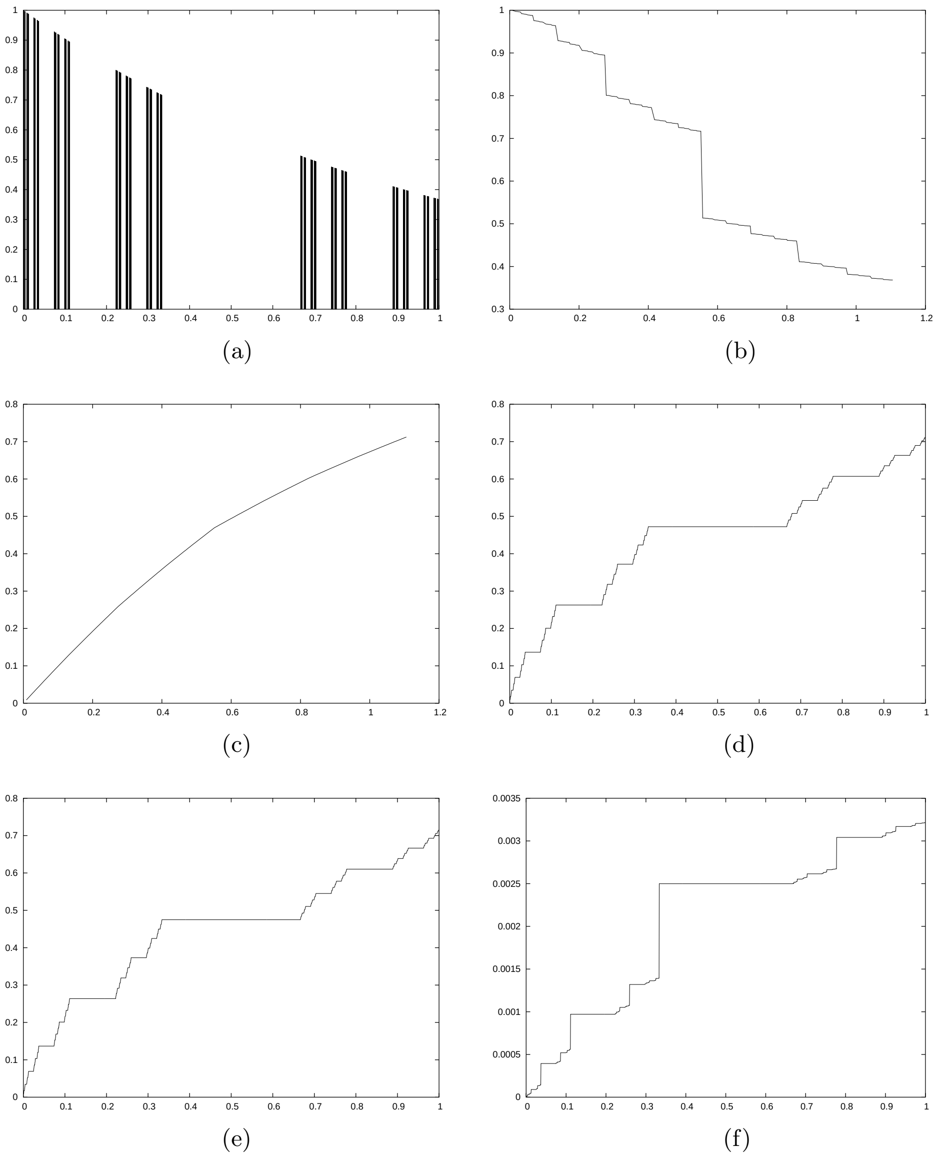

Figure 4. Numerical calculation of F^α-integral of $\exp(-x)\,\chi_C(x)$ where C is the middle $\frac{1}{3}$ Cantor set. (**a**) The function $f(x) = \exp(-x)\,\chi_C(x)$, (**b**) the function $g = \psi[f]$ (the vertical lines indicate discontinuities), (**c**) indefinite Riemann integral: $g_1(y) = \int_0^y \psi[f](y')\,\mathrm{d}y'$, (**d**) $\phi^{-1}[g_1](x)$, (**e**) $\int_0^x f(x')\,\mathrm{d}_F^\alpha x'$ calculated directly using the definition of F^α-integral, (**f**) the difference between (**d**) and (**e**), or equivalently the difference $(I_F^\alpha - \phi^{-1} I \psi) f(x)$, which vanishes in the limit.

$$\phi[\mathcal{D}_F^\alpha y] = \frac{\mathrm{d}}{\mathrm{d}t}\phi[y] = \frac{\mathrm{d}g}{\mathrm{d}t}.$$

So applying ϕ on both sides of (24), we get the defractalized equation

$$\frac{\mathrm{d}g}{\mathrm{d}t} = 1$$

which has a solution

$$g(t) = t \tag{25}$$

with the initial condition $g(t=0) = 0$. Now applying ϕ^{-1} to both sides of (25), we get

$$y(x) = S_F^\alpha(x)$$

which can be verified to be a solution to (24).

We now present another example. It is well-known that various relaxation phenomena in glassy materials, such as mechanical, dielectric or magnetic relaxation, are fractal time processes [35]. Let $\mu(t)$ denote the corresponding quantity (such as the polarization or magnetic moment).

We propose a model for such processes which involves F^α-calculus. We assume that $\mathrm{Sch}(\mu) \subset F$, i.e. $\mu(t)$ 'changes' only on a fractal subset F of time, F is α-perfect for some $\alpha \in (0,1]$, and the 'fractal rate' of change of $\mu(t)$ (i.e. the F^α-derivative) is proportional to $\mu(t)$ itself (with a negative sign). This can be written as

$$\mathcal{D}_F^\alpha \mu(t) = -k\mu(t)\chi_F(t), \tag{26}$$

where $k > 0$ is a proportionality constant. Let $y = \phi[\mu]$. Then $y(u) = \mu(t)$ for $u = S_F^\alpha(t)$. Further, by Theorem 18 in Appendix A, $\mathrm{d}y/\mathrm{d}u = \mathcal{D}_F^\alpha \mu(t)$. Therefore, (26) transforms into the 'defractalized' equation

$$\frac{\mathrm{d}y}{\mathrm{d}u} = -ky \tag{27}$$

which has a solution

$$y = A\exp(-ku). \tag{28}$$

Applying ϕ^{-1} to y, we obtain a solution of the F^α-differential equation (26):

$$\mu(t) = A\exp(-kS_F^\alpha(t)) \tag{29}$$

with the initial condition $\mu(0) = A$. This solution evolves on the fractal F. Further, for many sets F, like the Cantor sets, it is known that S_F^α is bounded above and below by constant multiples of t^α. It is empirically known [35] that the relaxation function defined by $w(t) = \langle \mu(t)\mu(0)\rangle / \langle \mu^2(0)\rangle$ has the form

$$w(t) = \exp[-(t/\tau)^\alpha], \quad 0 < \alpha < 1. \tag{30}$$

Thus, (26) is a plausible model for such processes, involving local differentiation operator. More importantly, we have demonstrated the use of the defractalizing map ϕ in obtaining its solution.

6. Concluding remarks

Calculus on fractal subsets of real line, or F^α-calculus [31], is a calculus suitable, among other themes, to model fractal time processes, or sublinear behaviours. In this paper, we have shown the use of the F^α-calculus in formulating and solving fractal-time differential equations.

The F^α-differential equations (of which fractal-time differential equations are a special case) are differential equations involving the F^α-derivatives exactly like the ordinary differential equations involving the ordinary derivatives. As demonstrated in §3–5 they offer possibilities of modeling dynamical behaviours naturally for which ordinary differential equations and methods of ordinary calculus are inadequate.

Continuous-time dynamical systems are associated with ordinary differential equations, and discrete-time dynamical systems are associated with maps/diffeomorphisms. But the dynamical systems associated with F^α-differential equations are a new class of dynamical systems [21,22,31] which evolve on fractal subsets of time-axis.

We have discussed simple models based on F^α-differential equations. The solutions of F^α-differential equations naturally involve staircase-like functions. Staircase functions such as the Lebesgue–Cantor staircase function are known to be bounded by sublinear power laws. Also, they 'change' or 'evolve' only on a fractal set. Thus, this framework may be useful in modeling many cases of sublinear behaviour, fractal time evolution, fields due to fractal charge distributions, etc.

The fractalizing and defractalizing transformations, sketched in §4 and detailed in Appendix A, induce a conjugacy between F^α-integral (F^α-derivative) and Riemann integral (ordinary derivative). The defractalizing transformation takes an F^α-integrable (F^α-differentiable) function f to a Riemann integrable (ordinarily differentiable) function g, such that the corresponding types of integrals (derivatives) have equal values.

One of the important uses of this conjugacy is in solving F^α-differential equations. The method of defractalizing fractal equations to obtain ordinary differential equations, solving them and fractalizing the solution is a practical method to construct solutions of fractal differential equations. This is demonstrated with the help of an example in which we have considered a model for fractal-time relaxation processes.

There are many interesting questions in the theme of F^α-differential equations that need to be addressed such as fractal variational principles, classification of sublinear behaviours, multidimensional F^α-calculus, F^α-differential equations involving distributions, etc. Work is in progress in these directions.

Appendix A: Conjugacy between F^α-calculus and ordinary calculus

In this appendix we briefly state some results concerning relations between ordinary calculus and F^α-calculus developed in [32].

A.1 *Conjugacy between F^α-integral and Riemann integral*

First we introduce relevant classes of functions. Let $F \subset \mathbf{R}$ be an α-perfect set for some $\alpha \in (0,1]$. Let $[a,b]$ be a closed interval. Then

1. K denotes the range of S_F^α. K is of the form of a bounded or unbounded interval.
2. $B(F)$ denotes the class of functions bounded on F.
3. $B(K)$ denotes the class of functions bounded on K.
4. $\tilde{B}(F)$ denotes the class of functions $h \in B(F)$ such that

   $$x_1, x_2 \in F \quad \text{and} \quad S_F^\alpha(x_1) = S_F^\alpha(x_2) \Longrightarrow h(x_1) = h(x_2).$$

   For example, let $F \subset [a,b]$. Then, the function $f_1(x) = \chi_F(x) S_F^\alpha(x)$ belongs to $\tilde{B}(F)$. On the other hand, the function $f_2(x) = x\chi_F(x)$ belongs to $B(F)$ but not $\tilde{B}(F)$.
5. $\mathcal{F}$ denotes the class of functions $f \in B(F)$ which are F^α-integrable over $[a,b]$.
6. $\mathcal{H}$ denotes the class of functions $h \in \mathcal{F}$ such that

   $$x_1, x_2 \in F \quad \text{and} \quad S_F^\alpha(x_1) = S_F^\alpha(x_2) \Longrightarrow h(x_1) = h(x_2),$$

   i.e. $\mathcal{H} = \mathcal{F} \cap \tilde{B}(F)$.
7. $\mathcal{G}$ denotes the class of functions in $B(K)$ which are Riemann integrable over the interval $[S_F^\alpha(a), S_F^\alpha(b)]$.

Now we proceed to define the required maps and state the results. We begin by defining the first defractalizing map ϕ.

DEFINITION 9

The map ϕ: $\tilde{B}(F) \to B(K)$ takes $h \in \tilde{B}(F)$ to $\phi[h] \in B(K)$ such that for each $x \in F$,

$$\phi[h](S_F^\alpha(x)) = h(x).$$

Theorem 10. *The map $\phi : \tilde{B}(F) \to B(K)$ is one-to-one and onto.*

The map ϕ takes a function $h \in \tilde{B}(F)$ to $g \in B(K)$ such that the F^α-integral of h is equal to the Riemann integral of g on corresponding intervals (if either exists).

Theorem 11. *A function $h \in \tilde{B}(F)$ is F^α-integrable over $[a,b]$ if and only if $g = \phi[h]$ is Riemann integrable over $[S_F^\alpha(a), S_F^\alpha(b)]$. In other words, a function $h \in \tilde{B}(F)$ belongs to $\mathcal{H}$ if and only if g belongs to $\mathcal{G}$. Further in that case,*

$$\int_a^b h(x)\,\mathrm{d}_F^\alpha x = \int_{S_F^\alpha(a)}^{S_F^\alpha(b)} g(u)\,\mathrm{d}u.$$

As an example, consider the integral

$$\int_0^b h(x)\,\mathrm{d}_F^\alpha x, \qquad h(x) = (S_F^\alpha(x))^n, \qquad S_F^\alpha(0) = 0.$$

Clearly, $h \in \tilde{B}(F)$, and further, $h \in \mathcal{H}$. Now if $g = \phi[h]$, then $g(u) = u^n$. Thus, by Theorem 11,

$$\int_0^b h(x)\,\mathrm{d}_F^\alpha x = \int_0^{S_F^\alpha(b)} g(u)\,\mathrm{d}u = \frac{1}{n+1}(S_F^\alpha(b))^{n+1}$$

which agrees with the result obtained in [31] using first principles.

To extend the domain of ϕ from $\tilde{B}(F)$ to $B(F)$, we now define an intermediate map η which takes $f \in B(F)$ to $h \in \tilde{B}(F)$ preserving F^α-integrals.

DEFINITION 12

The map η: $B(F) \to \tilde{B}(F)$ is defined as

$$\eta[f](x) = \begin{cases} \min_{\{z \in F: S_F^\alpha(z) = S_F^\alpha(x)\}} f(z) & \text{if } x \in F \\ 0 & \text{if } x \notin F. \end{cases}$$

This construction is such that for $f \in B(F)$, $\eta[f]$ is the largest function in $\tilde{B}(F)$ which is nowhere greater than f on F. Further, if $f \in \tilde{B}(F)$, then $\eta[f] = f$, i.e., η restricted to $\tilde{B}(F)$ is identity. F^α-integrals are preserved under η.

Theorem 13. *Let the function $f \in B(F)$ be F^α-integrable over $[a,b]$. Then $h = \eta[f]$ is F^α-integrable over $[a,b]$ and*

$$\int_a^b f(x)\,d_F^\alpha x = \int_a^b h(x)\,d_F^\alpha x.$$

In other words, η preserves F^α-integrals.

Now we introduce the composite map ψ which extends the domain of ϕ (because η is an identity over $\tilde{B}(F)$).

DEFINITION 14

The map ψ: $B(F) \to B(K)$ is the composite map $\phi \circ \eta$.

This second defractalizing map ψ takes $f \in B(F)$ to $g \in B(K)$ such that if f is F^α-integrable, then g is Riemann integrable and the respective integrals are equal.

Theorem 15. *Let $f \in \mathcal{F}$, and let $g = \psi[f] = \phi \circ \eta[f]$. Then $g \in \mathcal{G}$ (i.e. it is Riemann integrable over $[S_F^\alpha(a), S_F^\alpha(b)]$) and*

$$\int_a^b f(x)\,\mathrm{d}_F^\alpha x = \int_{S_F^\alpha(a)}^{S_F^\alpha(b)} g(u)\,\mathrm{d}u.$$

A.2 *Conjugacy between the F^α-derivative and the ordinary derivative*

In this section, we show that the same map ϕ also establishes the conjugacy between the F^α-derivative and the ordinary (first order) derivative under certain conditions.

The next theorem enables us to calculate the F^α-derivative of a function h if the ordinary derivative of its image $\phi[h]$ is known.

Theorem 16. *Let h be a function in $\tilde{B}(F)$ and let $g = \phi[h]$ be differentiable on $[S_F^\alpha(a), S_F^\alpha(b)]$. Then $\mathcal{D}_F^\alpha(h(x))$ exists, belongs to $\tilde{B}(F)$, and*

$$\mathcal{D}_F^\alpha h(x) = \frac{\mathrm{d}g(t = S_F^\alpha(x))}{\mathrm{d}t}$$

for all $x \in F \cap [a,b]$.

As an example, let us consider the function $h(x) = (S_F^\alpha(x))^n$, $n = 1, 2, \ldots$. Clearly, $h \in \tilde{B}(F)$. Let $g = \phi[h]$. Then $g(t) = t^n$. Now according to Theorem 16, for $x \in F$,

$$\mathcal{D}_F^\alpha(h(x)) = \frac{\mathrm{d}g(t = S_F^\alpha(x))}{\mathrm{d}t} = nt^{n-1}\Big|_{t=S_F^\alpha(x)} = n(S_F^\alpha(x))^{n-1}.$$

This result agrees with the one obtained in [31] using first principles.

The next theorem enables us to express the ordinary derivative of $g = \phi[h]$ in terms of the F^α-derivative of h. This is one step towards the converse of Theorem 16. However, we note, intuitively, that because of the 'fractured' nature of a typical fractal F, F-limits at some points of F correspond only to one sided limits under ϕ. Therefore at such points, only one-sided ordinary derivatives are guaranteed.

Theorem 17. *Let $h \in \tilde{B}(F)$ be an F^α-differentiable function on $[a,b]$ and let $g = \phi[h]$.*

1. *If for $x \in F$ there exists $y \in F$, $y < x$ such that $S_F^\alpha(y) = S_F^\alpha(x)$ $(= \tau$, say$)$, then $\mathrm{d}g/\mathrm{d}t_-$ and $\mathrm{d}g/\mathrm{d}t_+$ exist at $t = S_F^\alpha(x)$ and*

$$\frac{\mathrm{d}g(t = \tau)}{\mathrm{d}t}\Big|_- = \mathcal{D}_F^\alpha h(y),$$

and

$$\frac{\mathrm{d}g(t = \tau)}{\mathrm{d}t}\Big|_+ = \mathcal{D}_F^\alpha h(x),$$

where the subscripts $-$ and $+$ of $\mathrm{d}g/\mathrm{d}t$ denote the left- and the right-handed derivatives respectively.

2. *If $x \in F$ is such that for every $y \in [a,b]$, $y \neq x \Longrightarrow S_F^\alpha(y) \neq S_F^\alpha(x)$, then $\mathrm{d}g/\mathrm{d}t$ exists at $t = S_F^\alpha(x)$ and*

$$\frac{\mathrm{d}g(t = S_F^\alpha(x))}{\mathrm{d}t} = \mathcal{D}_F^\alpha h(x).$$

For certain kinds of functions h, the left and right derivative of $g = \phi[h]$ are always equal and therefore can be replaced by the derivative of g. Thus we come to the converse of Theorem 16.

Theorem 18. *Let $h \in \tilde{B}(F)$ be an F^α-differentiable function on $[a,b]$ such that $\mathcal{D}_F^\alpha h \in \tilde{B}(F)$. Let $g = \phi[h]$. Then for all $x \in F \cap [a,b]$,*

$$\frac{\mathrm{d}g(t = S_F^\alpha(x))}{\mathrm{d}t} = \mathcal{D}_F^\alpha h(x). \tag{A1}$$

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References

[1] B B Mandelbrot, *The fractal geometry of nature* (Freeman and Company, 1977)
[2] A Bunde and S Havlin (Eds), *Fractals in science* (Springer, 1995)
[3] B J West, M Bologna and P Grinolini, *Physics of fractal operators* (Springer Verlag, New York, 2003)
[4] K Falconer, *The geometry of fractal sets* (Cambridge University Press, 1985)
[5] K Falconer, *Fractal geometry: Mathematical foundations and applications* (John Wiley and Sons, 1990)
[6] K Falconer, *Techniques in fractal geometry* (John Wiley and Sons, 1997)
[7] G A Edgar, *Integral, probability and fractal measures* (Springer-Verlag, New York, 1998)
[8] S G Samko, A A Kilbas and O I Marichev, *Fractional integrals and derivatives – Theory and applications* (Gordon and Breach Science Publishers, 1993)
[9] R Hilfer, *Applications of fractional calculus in physics* (World Scientific Publ. Co., Singapore, 2000)
[10] K S Miller and B Ross, *An introduction to the fractional calculus and fractional differential equations* (John Wiley, New York, 1993)
[11] K B Oldham and J Spanier, *The fractional calculus* (Academic Press, New York, 1974)
[12] R Metzler, W G Glöckle and T F Nonnenmacher, *Physica* **A211**, 13 (1994)
[13] R Metzler, E Barkai and J Klafter, *Phys. Rev. Lett.* **82**, 3563 (1999)
[14] R Hilfer and L Anton, *Phys. Rev.* **E51**, R848 (1995)
[15] A Compte, *Phys. Rev.* **E53**, 4191 (1996)
[16] G M Zaslavsky, *Physica* **D76**, 110 (1994)
[17] R Metzler, E Barkai and J Klafter, *Physica* **A266**, 343 (1999)
[18] R Hilfer, *J. Phys. Chem.* **B104**, 3914 (2000)
[19] K M Kolwankar and A D Gangal, *Chaos* **6**, 505 (1996)
J Levy Vehel and K M Kolwankar, *Fract. Calc. Appl. Anal.* **4**, 285 (2001)
[20] K M Kolwankar and A D Gangal, *Pramana – J. Phys.* **48**, 49 (1997)
[21] K M Kolwankar and A D Gangal, *Phys. Rev. Lett.* **80**, 214 (1998)
[22] K M Kolwankar and A D Gangal, Local Fractional Calculus: A Calculus for Fractal Space-Time, in: *Fractals: Theory and applications in engineering* edited by M Dekking, J Levy Vehel *et al* (Springer, London, 1999)
[23] F B Adda and J Cresson, *J. Math. Anal. Appl.* **263**, 721 (2001)
[24] A Babakhani and V Daftardar-Gejji, *J. Math. Anal. Appl.* **270**, 66 (2002)

[25] M T Barlow, *Diffusion on fractals, Lecture notes* (Math. Vol. 1690, Springer, 1998)
[26] J Kigami, *Analysis on fractals* (Cambridge University Press, 2000)
[27] K Dalrymple, R S Strichartz and J P Vinson, *J. Fourier Anal. Appl.* **5**, 205 (1999)
[28] R S Strichartz, *J. Funct. Anal.* **174**, 76 (2000)
[29] U Freiberg and M Zähle, *Potential Anal.* **16**, 265 (2002)
[30] U Freiberg and M Zähle, Harmonic calculus on fractals – A measure geometric approach II (2000) Preprint
[31] A Parvate and A D Gangal, math-ph/0310047 (2003)
[32] A Parvate and A D Gangal, *Calculus on fractal subsets of real line – II: Conjugacy with ordinary calculus*, Pune University Preprint (2004)
[33] R R Goldberg, *Methods of real analysis* (Oxford and IBH Publishing Co. Pvt. Ltd., 1970)
[34] E Hille and J D Tamarkin, *American Mathematics Monthly* **36**, 255 (1929)
[35] M F Shlesinger, *Ann. Rev. Phys. Chem.* **39**, 269 (1988)